EXAMPLES OF A FEW ELEMENTARY GROUPS.*

BY DR. G. A. MILLER.

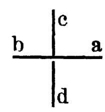
It is easy to verify that the following four substitutions,

$$1 \qquad ab \qquad cd \qquad ab.cd$$

constitute a substitution group, which has the following properties: (1) The product of any two of these substitutions of order two is the third, and the product of the three is identity; (2) These products are independent of the order of the factors, *i. e.*, all the substitutions are commutative;† (3) The smallest group that contains two of these substitutions of order two must also contain the third, *i. e.*, the group is generated by any two of its substitutions of order two, but it is not generated by one of them; (4) The group contains three subgroups of order two and one of order one.

We proceed to give several geometric illustrations of this group.

Representing the positive half of the x-axis by a, the negative half by b, the positive half of the y-axis by c, and the negative half by d, we observe that the rotation of the plane around the y-axis through an angle of 180° corresponds to the substitution ab, and the rotation around the x-axis through 180° corresponds to cd. The effect obtained by these two rotations, in succession, is clearly equivalent to a rotation of the plane through 180° on the origin as a pivot. This operation is also of order two and it corresponds to $ab.cd$.

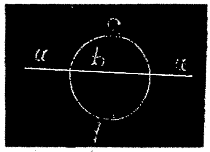


Since the law of combination of these three operations is exactly the same as that of the given substitutions we say that these three operations and identity constitute a group which is *simply isomorphic* to the given substitution group. Representing the given operations analytically we obtain the following equations:

$$ab\begin{cases} x'=-x \\ y'=y \end{cases} \qquad cd\begin{cases} x'=x \\ y'=-y \end{cases} \qquad ab.cd\begin{cases} x'=-x \\ y'=-y \end{cases}$$

It may be observed that the third of these operations is equivalent to a rotation of space around the z-axis through 180°.

We may obtain another geometric illustration of the same group by considering the inversion of the plane with respect to any circle of radius k and the rotation of the plane through an angle of 180° around a line passing through the



*In this article we shall not presuppose any knowledge of the theory of groups except the facts which were developed in our article in the November number of this Journal.

†If every two substitutions of a group are commutative the group is said to be Abelian. Hence this group of order four is an Abelian group. The group of order six which is given in the said article of the November number of this Journal is non-Abelian.

center of the circle. Since each of these operations is of order two and since they are commutative their product must be of order two and it must be commutative with each of these two operations. For, if s_1, s_2 represent two different commutative operations of order two we have

$$(s_1 s_2)^2 = s_1 s_2 s_1 s_2 = s_1^2 s_2^2 = 1.$$

This proves that $s_1 s_2$ is of order two. From $s_1 s_2 s_1 = s_1 s_1 s_2$ we observe that s_1 is commutative with $s_1 s_2$. Similarly we see that s_2 is commutative with $s_1 s_2$. Hence we observe that *any two different commutative operations of order two must generate a group which is simply isomorphic to the given group of order four.* If we use the given line as the x-axis and the perpendicular to it through the center of the circle as the y-axis we may represent the given operations analytically as follows :

$$\text{*}\ ab\begin{cases} x' = \dfrac{k^2 x}{x^2+y^2} \\ y' = \dfrac{k^2 y}{x^2+y^2} \end{cases} \qquad cd\begin{cases} x' = x \\ y' = -y \end{cases} \qquad ab.cd\begin{cases} x' = \dfrac{k^2 x}{x^2+y^2} \\ y' = -\dfrac{k^2 y}{x^2+y^2} \end{cases}$$

To verify analytically that the last one of these operations is of order two we let

$$x'' = \frac{k^2 x'}{x'^2 + y'^2} = k^4 \frac{x}{x^2+y^2} \frac{(x^2+y^2)^2}{k^4(x^2+y^2)} = x,$$

$$y'' = -\frac{k^2 y'}{x'^2 + y'^2} = k^4 \frac{y}{x^2+y^2} \frac{(x^2+y^2)^2}{k^4(x^2+y^2)} = y.$$

We have now given two geometric illustrations of the given group of order four† and we observed that the characteristic property of this group is, that it is generated by two different commutative operations of order two. There is another substitution group of order four whose characteristic property is entirely different. The substitutions of this group are

$$1 \qquad ac.bd \qquad abcd \qquad adcb.$$

Each of the last two substitutions is the third power of the other and each one of these generates the entire group ; *i. e.*, the smallest group that contains one of these substitutions must contain all the substitutions of this group of order four. It can readily be verified that these four substitutions obey the same com-

*These substitutions may be obtained by representing the segment of the axis which is outside the circle by a, the segment within the circle by b, the upper semi-circumference by c, and the lower semi-circumference by d.

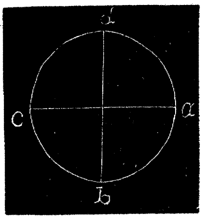
†In the latter example we could have inverted space with respect to a sphere intead of inverting the plane with respect to a circle.

binatory laws as the numbers which are written below them in the following arrangement:

1	*ac.bd*	*abcd*	*adcb*
1	-1	$\sqrt{-1}$	$-\sqrt{-1}$

The last two numbers can evidently be interchanged without affecting the laws of combination, but none of the other numbers permit such an interchange.

If we denote the points where two perpendicular diameters meet a circle by a, b, c, d, we observe that the substitution *abcd* is equivalent to rotating this circle on its center through 90°, *ac.bd* is equivalent to a rotation through 180°, and *adcb* is equivalent to a rotation through 270°, or through $-90°$. The characteristic property of this group is that it is generated by an operator* of order four. When a group is generated by a single operator of order n it is called the *cyclical* group of order n. It should be observed that the cyclical group of order four contains only one subgroup of order two, viz., the one which corresponds to the rotations through 180° and 360°, while the given non-cyclical group of this order contains three such subgroups.



We have now considered two groups of order four whose combinatory laws are different, *i. e.*, two groups which are not simply isomorphic. Such groups are said to be distinct *abstract* groups. Two groups which are simply isomorphic are said to be the same abstract group, regardless of the notation by means of which they may be represented, *e. g.*, 1, *ab*, *cd*, *ab.cd*, and 1, *ab.cd*, *ac.bd*, *ad.bc* are different as substitution groups but they represent the same abstract group since the law of combination of their substitutions is the same. We may state without proof that there are only two abstract groups of order four; *i. e.*, If four operators form a group their laws of combination must be the same as those of one of the given groups of order four.

We proceed to give a geometric illustration of the group of order six which is composed of all the substitutions that can be formed with three letters.† The substitutions of this group are

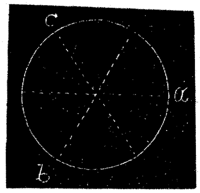
1 *abc* *acb* *ab* *ac* *bc*

Dividing the circle into three equal parts and drawing diameters through these points of division, we observe that *abc* and *acb* correspond to rotations of the circle on its center through 120° and 240°, respectively; *ab*, *ac*, and *bc* corres-

*The substitutions of a group represent operators as well as the result of operations. The group elements may therefore be called operations or operators. It is necessay to distinguish between the group elements and the elements of the substitutions, the former term is frequently used to denote the operators of a group since the group is really composed of these operators as elementary parts. When the word "element" is used in connection with a group we have sometimes to decide from the context whether it means an operator or a letter of the substitutions of the group.

†The group which is composed of all the possible substitutions of degree n is called the *symmetric* group of degree n. It is of order $n!$. Cf. THE AMERICAN MATHEMATICAL MONTHLY, Vol. VI, page 257.

pond to the rotations of the plane through 180° around the diameters going through c, b, and a, respectively. If we perform any two of these rotations in succession the result is equivalent to a single rotation which corresponds to the product of the two substitutions corresponding to the two rotations and taken in the same order ; *e. g.*, the rotation on the center through 240°, followed by rotation through 180° around the diameter through a is equivalent to the single rotation through 180° on the diameter through c, since $acb.bc=ab$. This result should also be seen geometrically.



All the products given in Vol. VI, page 256, of this Journal, may be directly verified by means of the last figure. The six rotations which correspond to the substitutions of the symmetric group of degree three are thus seen to form a very interesting group of rotations according to which the plane (or space) may be transformed. The determination of all the possible groups of motion by means of which space may be transformed forms a very interesting problem in the theory of groups, which was first studied by Camille Jordan, *Annali di Matematiche*, 1868, Vol. 2, page 167. The group of finite rotations are given in somewhat greater details in Klein's *Ikosaeder*, 1884, Chapter I.

Another important illustration of the symmetric group of three elements is furnished by the six anharmonic ratios of four points. These ratios may be placed in six different ways in a 1, 1 correspondence with the substitutions of this symmetric group. One of these ways is as follows :

1	abc	acb	ab	ac	bc
λ, λ	$\lambda, \frac{\lambda-1}{\lambda}$	$\lambda, \frac{1}{1-\lambda}$	$\lambda, 1-\lambda$*	$\lambda, \frac{1}{\lambda}$	$\lambda, \frac{\lambda}{\lambda-1}$

where the notation $\lambda, \frac{\lambda-1}{\lambda}$ means that λ is to be replaced by $\frac{\lambda-1}{\lambda}$. *E. g.*, performing the third and fourth operation in succession, we have $\lambda, \frac{1}{1-(1-\lambda)}=\lambda, 1/\lambda$ just as $acb.ab=ac$; performing the fourth and third operation in succession, we have $\lambda, 1-\frac{1}{1-\lambda}=\lambda, \frac{\lambda}{\lambda-1}$ just as $ab.acb=bc$, etc. For other illustrations of this group the reader may consult Burnside's *Theory of Groups*, 1897, page 18.

The symmetric group of degree four contains 24 substitutions. These correspond to the 24 rotations which transform a cube into itself, for these rotations permute the four diagonals of the cube in every possible manner. The axes of rotation are the lines which join the middle points of the opposite faces, those which join the middle points of the opposite edges, and the diagonals. There are three axes of the first kind and we may rotate the cube around one of these

*It may be observed that $(\lambda, 1-\lambda)^2=1$ while $(\lambda, \lambda-1)^\alpha=\lambda, \lambda-\alpha$; *i. e.*, the first of these two substitutions is of order two while the second does not have a finite order.

axes through an angle of 90°, 180°, 270°, or 360° so that after each rotation the entire cube occupies the same space as it did before the rotation. Hence the symmetric group of degree four contains three cyclical subgroups of order four. Each of these contains a subgroup of order two and no two of these subgroups of order two are identical.

There are six axes of the second kind and we may rotate the cube into itself around one of these axes through 180° or 360°. As the corresponding subgroups of order two are different from the three given above we observe that the symmetric group of degree four contains nine subgroups of order two. The rotations around the diagonals correspond to the four snbgroups of order three that are contained in the symmetric group of degree four. We have now employed all the possible rotations which transform the cube into itself without changing its center, and have seen that the corresponding permutations of the diagonals give all the possible substitutions that can be formed with four elements. These 24 rotations constitute an interesting group of motion.

It is easy to see that the different powers of a circular substitution of degree n $(a_1 a_2 a_3 \ldots\ldots a_{n-1} a_n)$ constitute a group of order n. When $n=3$ we have the substitutions 1, $a_1 a_2 a_3$, $a_1 a_3 a_2$, and when $n=4$, the substitutions 1, $a_1 a_3 \cdot a_2 a_4$, $a_1 a_2 a_3 a_4$, $a_1 a_4 a_3 a_2$. These groups have been considered. In general we may divide the circumference of a circle into n equal parts and represent the points of division by $a_1, a_2, a_3, \ldots \ldots, a_n$. The n different positive rotations around the center of the circle through angles which are divisible by $2\pi/n$ will clearly constitute a group of operations that is simply isomorphic to the substitution group generated by the given circular substitution. Since the equation $x^n - 1 = 0$ has primitive roots all the roots of this equation constitute a group which is simply isomorphic with the cyclical group of order n.

From the preceding examples it may be inferred that the same group may present itself in many different forms as well as in different branches of mathematics. The fundamental group concept is that there is a system of operations (substitutions, rotations, complex numbers, etc.) such that the product of any two of them and the square of any one are again in the system. This necessary condition is not always a sufficient condition that a system of operations may constitute a group, but many operations, such as substitutions, obey *per se* the other necessary conditions.*

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*Cf. Burnside, *Theory of Groups of a Finite Order*, page 11, or Weber's *Lehrbuch der Algebra*, Vol. 2, page 2.